On Flavors of Factorization in Commutative Rings with Zero Divisors

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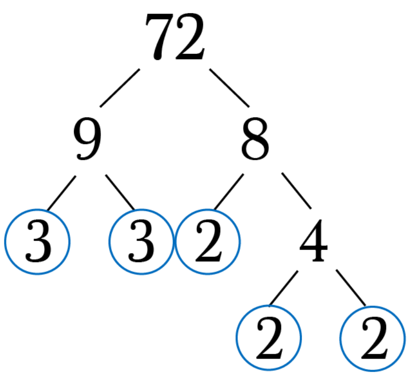
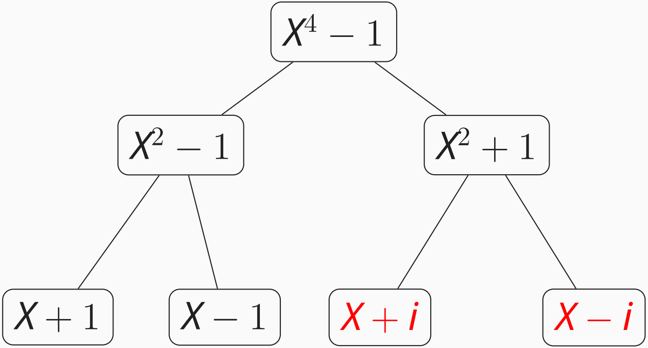
What is Factorization Theory?

Fundamental Theorem of Arithmetic



FTA: Every integer greater than 1 can be factored uniquely as the product of primes

Unique Factorization



Unique factorization depends on the setting!

Unique Factorization

$a \in R$ is an **atom** if $a = bc$ implies that,
(1) b or c is a unit,
(2) either $a \mid b$ and $b \mid a$ or $a \mid c$ and $c \mid a$, i.e. a is **associated** to b or a is **associated** to c.

ex. $X + 1 = \frac{1}{2}(2X + 2)$ in $\mathbb{R}[X]$ but $X + 1 \sim 2x + 2$

unique factorization domain (UFD): every element can be factored uniquely into the product of atoms up to order and associates

ex. $\mathbb{Z}, \mathbb{Z}[i], \mathbb{R}, \mathbb{C}$

Unique Factorization in $R[X]$

Theorem: R is a UFD if and only if $R[X]$ is a UFD

ex. $\mathbb{Z}[X], (\mathbb{Z}[i])[X], \mathbb{R}[X], \mathbb{C}[X], \mathbb{C}[X, Y, Z]$.

Non-Unique Factorization

Consider the ring: $\boxed{\mathbb{R} + X\mathbb{C}[X]}$

<u>in</u>

- $\sqrt{3} + X(2iX^3 + i)$
- X
- $\left(\frac{1+i}{2}\right)X$

<u>out</u>

- $3i$
- $1 + i$

factorization of X^2 in $\mathbb{R} + X\mathbb{C}[X]$

$$\begin{aligned} X^2 &= X \cdot X \\ &= (iX)(-iX) \\ &= (1+i)X\left(\frac{1-i}{2}\right)X \\ &= \underbrace{(2+i)X\left(\frac{2-i}{5}\right)X}_{X^2 \text{ is divisible by } \{(r+i)X\}} \end{aligned}$$

Non-unique Factorization

half-factorial domain (HFD): every factorization of a nonzero nonunit element into atoms has the same length

examples: $\mathbb{R} + X\mathbb{C}[X]$, $\mathbb{Z}\sqrt{-5}$, any UFD

Non-unique Factorization

Consider the ring $\mathbb{R}[X^2, X^3]$,

$$X^6 = \underbrace{X^2 \cdot X^2 \cdot X^2}_{\text{length 3}} = \underbrace{X^3 \cdot X^3}_{\text{length 2}}$$

X^6 has two nonassociate factorizations into atoms of different lengths!

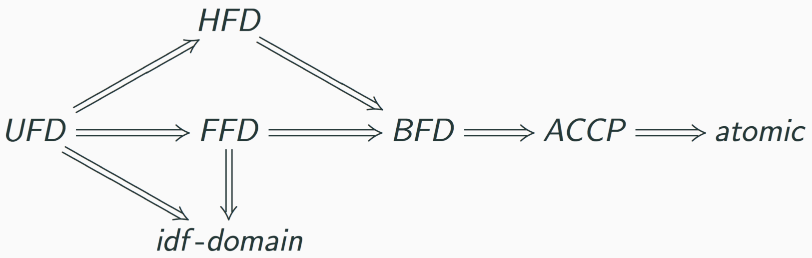
Non-unique Factorization

finite factorization domain (FFD): every factorization of a nonzero nonunit element into atoms has finite length

examples: $\mathbb{R}[X^2, X^3]$, $\mathbb{Z}\sqrt{-5}$, any UFD, some HFDs

Non-unique Factorization

(1990 Anderson et all) *Factorization in Integral Domains*



What are Zero Divisors?

Zero Divisors in Commutative Rings

an element $a \in R$ is a **zero divisor** if there is a nonzero element $b \in R$ so that $ab = 0$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Note: $2, 3$ and 4 are zero divisors in $\mathbb{Z}/6\mathbb{Z}$

Some Interesting Examples

Look at factorizations of 3 in $\mathbb{Z}/6\mathbb{Z}$.

$$
\begin{aligned}
&3 = 3 \\
&3 = 3 \cdot 3 \\
&3 = 3 \cdot 3 \cdot 3 \\
&\vdots \\
&3 = 3^n
\end{aligned}
$$

We have infinite factorizations in a finite ring!

Note: 3 is an **idempotent** element $(e^2 = e)$.

$$e^2 - e = 0 \Rightarrow e(e-1) = 0.$$

Some Interesting Examples

Look at the factorization of X in $(\mathbb{Z}/6\mathbb{Z})[X]$,

$$(3X+2)(2X+3) = 6X^2 + 13X + 6 = X$$

Intuitive degree arguments fail!

Some Interesting Examples

Factorizations of powers of X in $(\mathbb{Z}/4\mathbb{Z})[X]$

$$X^2 = X \cdot X = (X+2)(X+2)$$

$$X^3 = X \cdot X \cdot X = X(X+2)(X+2)$$

$$X^4 = X \cdot X \cdot X \cdot X = (X^2+2)(X^2+2)$$

$$X^5 = X \cdot X \cdot X \cdot X \cdot X = X(X^2+2)(X^2+2)$$

Polynomial Rings

Question: If R is a unique factorization ring with zero divisors, does $R[X]$ have the unique factorization property?

Not necessarily: $\mathbb{Z}/4\mathbb{Z}$ is a UFR (not a UFD) but $(\mathbb{Z}/4\mathbb{Z})[X]$ is not a UFR

$$X^2 = X \cdot X = (X+2)(X+2)$$

Working With Zero Divisors

Approach 1: Only worry about the **regular** elements

ex. R is **factorial** if every regular element factors uniquely as the product of atoms.

Note: $R[X]$ is factorial if and only if R is a UFD.

Working With Zero Divisors

Approach 2: "weaken" properties from integral domains and then generalize

reduced	$a \neq a_1 \cdots \hat{a}_i \cdots a_n$ for any $i \in \{1, \ldots, n\}$
strongly reduced	$a \neq a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_j} \cdots a_n$ for any nonempty proper subset $\{i_1, \cdots, i_j\} \subsetneq \{1, \ldots, n\}$.

ex. $(1,0) = (2,0)(\frac{1}{2},0)(2,0)(\frac{1}{2},0)$ in $\mathbb{Q} \times \mathbb{Q}$

is reduced but NOT strongly reduced

Note: $(1,0)$ is an idempotent in $\mathbb{Q} \times \mathbb{Q}$

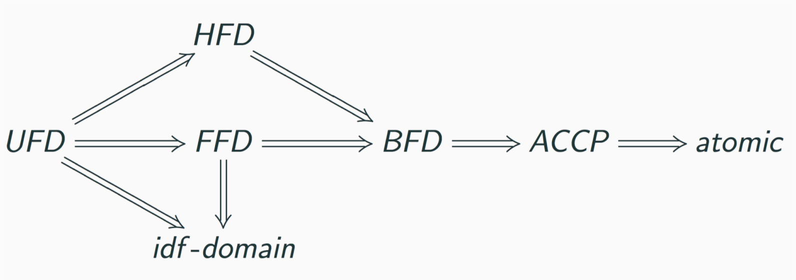
Reduced UFRs

R is a **strongly reduced** (respectively **reduced**) UFR if:
1) R is atomic
2) if $a = a_1 \cdots a_n = b_1 \cdots b_m$ are two strongly reduced (respectively reduced) factorizations of a nonunit $a \in R$, then $n = m$ and after a reordering $a_i \sim b_i$ for $i \in \{1, \dots, n\}$

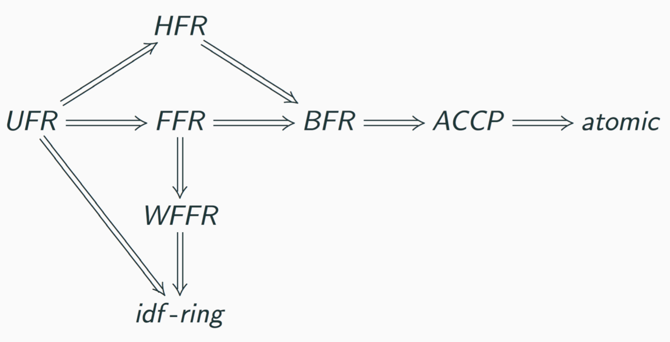
ex. $(\mathbb{Z}/6\mathbb{Z})[X]$ is a strongly reduced UFR

Note: the only strongly reduced factorization of 3 is $3 \cdot 1$

What are the benefits of "weakening" properties from domains to rings with zero divisors?



Property	UFD	HFD	FFD	idf	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

Types of UFRs

1. Fletcher UFR (1969)
2. Bouvier-Galovich UFR (1974-1978)
3. (α, β)-UFR (1996)
4. Reduced UFR (2003)
5. **Weak UFR** (2011) (ex. $(\mathbb{Z}/4\mathbb{Z})[X]$)

Note: Other properties outside of unique factorization have also been investigated.

ex. $\mathbb{Z}/6\mathbb{Z}$ is a reduced UFR and $(\mathbb{Z}/6\mathbb{Z})[X]$ is a reduced UFR

ex. $\mathbb{Z}/4\mathbb{Z}$ is a weak UFR and $(\mathbb{Z}/4\mathbb{Z})[X]$ is a weak UFR

Other Settings

Factorization in monoid rings $R[X, M]$

"polynomials" in X with coefficients in R and exponents in M

ex. $\mathbb{Z}[X; \mathbb{Z}/2\mathbb{Z}]$ is no longer a domain
$(X+1)(X-1) = X^2 - 1 = 1 - 1 = 0$

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