Unique Factorization in Polynomial Rings with Zero Divisors

Ranthony A.C. Edmonds
PhD Candidate
University of Iowa
Department of Mathematics

FACTORIZATION PROPERTIES

Definition

Two elements $a, b \in D$ where D is an integral domain are ***associated***, denoted $a \sim b$ if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$

Definition

An element $a \in D$ where D is an integral domain is ***irreducible*** if $a = bc$ implies $b \in U(R)$ or $c \in U(R)$

Theorem

In an integral domain the following are equivalent:

1. a is irreducible
2. $a = bc$ implies $a \sim b$ or $a \sim c$
3. (a) is maximal in the set of proper principal ideals of D

FACTORIZATION PROPERTIES

Definition

- a is ***associated*** to b in R, $a \sim b$, if $(a) = (b)$
- a is ***strongly associated***, $a \approx b$, if $a = ub$ for some $u \in U(R)$
- a is ***very strongly associated***, $a \cong b$, if (1) $a \sim b$ and (2) $a = b = 0$ or $a = rb$ implies $r \in U(R)$

very strongly associated $\Longrightarrow$ strongly associated $\Longrightarrow$ associated

FACTORIZATION PROPERTIES

Definition

- a is ***irreducible*** if $a = bc$ implies $a \sim b$ or $a \sim c$
- a is ***strongly irreducible*** if $a = bc$ implies $a \approx b$ or $a \approx c$
- a is ***very strongly irreducible*** if $a = bc$ implies $a \cong b$ or $a \cong c$
- a is ***m-irreducible*** if (a) is maximal in the set of proper principal ideals

$$\begin{array}{ccccccc} & & & & & & \text{prime} \\ & & & & & & \Downarrow \\ \text{v.s. irreducible} & \Longrightarrow & m\text{-irreducible} & \Longrightarrow & \text{s. irreducible} & \Longrightarrow & \text{irreducible} \end{array}$$

FACTORIZATION PROPERTIES

Definition

- R is ***atomic*** if each nonzero nonunit $a \in R$ is a finite product of irreducible elements (atoms)
- R is ***strongly atomic*** each nonzero nonunit $a \in R$ is a finite product of strongly irreducible elements
- R is ***very strongly atomic*** each nonzero nonunit $a \in R$ is a finite product of very strongly irreducible elements
- R is ***m-atomic*** if each nonzero nonunit $a \in R$ is a finite product of m-irreducible elements
- R is ***p-atomic*** if each nonzero nonunit $a \in R$ is a finite product of prime elements

FACTORIZATION PROPERTIES

very strongly associated $\Longrightarrow$ strongly associated $\Longrightarrow$ associated

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prime $\Downarrow$

v.s. irreducible $\Longrightarrow$ *m*-irreducible $\Longrightarrow$ s. irreducible $\Longrightarrow$ irreducible

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p-atomic $\Downarrow$

v.s. atomic $\Longrightarrow$ *m*-atomic $\Longrightarrow$ s. atomic $\Longrightarrow$ atomic

FACTORIZATION PROPERTIES

Definition

- Two factorizations of a nonunit $a \in R$ into nonunits $a = a_1 \cdots a_n = b_1 \cdots b_m$ are ***isomorphic*** if $n = m$ and there exists a permutation $\sigma \in S_n$ such that $a_i \sim b_{\sigma(i)}$
- Two factorizations of a nonunit $a \in R$ into nonunits $a = a_1 \cdots a_n = b_1 \cdots b_m$ are ***strongly isomorphic*** if $n = m$ and there exists a permutation $\sigma \in S_n$ such that $a_i \approx b_{\sigma(i)}$
- Two factorizations of a nonunit $a \in R$ into nonunits $a = a_1 \cdots a_n = b_1 \cdots b_m$ are ***very strongly isomorphic*** if $n = m$ and there exists a permutation $\sigma \in S_n$ such that $a_i \cong b_{\sigma(i)}$

(α, β)-UNIQUE FACTORIZATION RINGS

Definition

- Let $\alpha \in \{$atomic, strongly atomic, very strongly atomic, *m*-atomic, *p*-atomic$\}$ and
- $\beta \in \{$isomorphic, strongly isomorphic, very strongly isomorphic$\}$, then
- R is an (α, β)-***unique factorization ring*** if (1) R is α and (2) any two factorizations of a nonzero, nonunit element into irreducible elements of the type used to define α are β.

(α, β)-UNIQUE FACTORIZATION RINGS

Definition

R is called a ***unique factorization ring*** if *R* is an $(\alpha, \beta)-$ unique factorization ring for some (and hence all) (α, β) (except $\alpha =$ *p*-atomic).

Other Unique Factorization Rings

- Bouvier UFR is an (*m*-atomic, isomorphic)-unique factorization ring
- Galovich UFR is a (very strongly atomic, strongly isomorphic)-unique factorization ring
- Fletcher UFR
- reduced UFR

INDECOMPOSABLE ELEMENTS IN $R[X]$

Definition

An element $f \in R[X]$ is ***indecomposable*** if $f = gh$ implies $g \approx_{R[X]} a$ or $h \approx_{R[X]} a$ for some $a \in R$

Example 4.2.5 Let $R = \mathbb{Z}[B, C]/(5B, BC, 2C)$ where B, C are indeterminates over $\mathbb{Z}$. Denote the image of B and C by b, c respectively, so we can write $R = \mathbb{Z}[b, c]$. Note that $10 = (2 + bX)(5 + cX)$ but $2 + bX$ and $5 + cX$ are not strongly associated to any $a \in R$.

PROPERTIES OF INDETERMINATES

Theorem

For a commutative ring R the following are equivalent:

(1) X is irreducible in $R[X]$

(2) X is indecomposable in $R[X]$

(3) R is indecomposable

PROPERTIES OF INDETERMINATES

Theorem

Let R be a commutative ring. Then X is a product of n atoms if and only if R is a direct product of n indecomposable rings.

Corollary

When X is a finite product of atoms, the factorization is unique up to order and associates.

FACTORING POWERS OF INDETERMINATES

Question: *Does X^n have unique factorization in $R[X]$?*

FACTORING POWERS OF INDETERMINATES

Question: *Does X^n have unique factorization in $R[X]$?*

Example: X^n does not have unique factorization in $\mathbb{Z}_4[X]$,

- $X^2 = X \cdot X = (X+2)(X+2)$
- $X^3 = X \cdot X \cdot X = X(X+2)(X+2)$
- $X^4 = X \cdot X \cdot X \cdot X = (X^2+2)(X^2+2)$
- $X^5 = X \cdot X \cdot X \cdot X \cdot X = X(X^2+2)(X^2+2)$

FACTORING POWERS OF INDETERMINATES

Let $L(X^n)$ and $l(X^n)$ represent the longest and shortest lengths of a factorization of X^n into atoms in $\mathbb{Z}_4[X]$ and $\rho(X^n) = L(X^n)/l(X^n)$

Theorem

In $\mathbb{Z}_4[x]$, $L(X^n) = l(X^n)$ if $n = 1$ and for $n > 1$ $L(X^n) = n$,

$$l(X^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases} \text{ and } \rho(X^n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n/3 & \text{if } n \text{ is odd} \end{cases}.$$

BOUVIER-GALOVICH UNIQUE FACTORIZATION RING

Characterization of Bouvier-Galovich UFR

Given a commutative ring R, R is a Bouvier-Galovich unique factorization ring if R satisfies one of the following:

(1) R is a unique factorization domain,

(2) R is a quasi-local with unique maximal ideal M where $M^2 = 0$, or

(3) R is a special principal ideal ring.

BOUVIER-GALOVICH UNIQUE FACTORIZATION RING

Characterization of Bouvier-Galovich UFR for $R[X]$

$R[X]$ is a Bouvier-Galovich UFR if and only if $R[X]$ is a UFD.

BOUVIER-GALOVICH UNIQUE FACTORIZATION RING

Characterization of Bouvier-Galovich UFR for $R[X]$

$R[X]$ is a Bouvier-Galovich UFR if and only if $R[X]$ is a UFD.

Proof.

$\rightarrow$ Let $a, b \in R$ such $ab = 0$ so that a and b are nonzero

$\rightarrow$ $X, X - a$, and $X - b$ are irreducible since R is indecomposable

$\rightarrow$ We have
$$\begin{aligned}(X-a)(X-b) &= X^2 - (a+b)X + ab \\ &= X^2 - (a+b)X \\ &= X(X-(a+b))\end{aligned}$$

$\rightarrow$ A contradiction, so R is a domain and $R[X]$ is a UFD

□

FLETCHER UNIQUE FACTORIZATION RING

Characterization of Fletcher UFR

Given a commutative ring R, R is said to be a Fletcher unique factorization ring if and only if it is the finite direct product of unique factorization domains and special principal ideal rings.

Characterization of Fletcher UFR for $R[X]$

For a commutative ring R, $R[X]$ is a Fletcher UFR if and only if it is the finite direct product of unique factorization domains.

Definition

We say that a commutative ring R is a ***factorial ring*** if every regular nonunit element of R is a product of (regular) irreducibles and this factorization is unique up to order and associates.

Characterization of a Fletcher UFR for $R[X]$

For a commutative ring R, the following are equivalent:

1. $R[X]$ is a Fletcher UFR
2. $R[X]$ is p-atomic
3. R is finite direct product of UFDs
4. $R[X]$ is factorial
5. every regular element of $R[X]$ is a product of principal primes

REDUCED RINGS

Definition

- In a commutative ring R, a factorization $a = a_1 \cdot a_n$ of a nonunit $a \in R$ is ***reduced*** if $a \neq a_1 \cdots \hat{a}_i \cdots a_n$ for any $i \in \{1, \ldots, n\}$
- In a commutative ring R, a factorization $a = a_1 \cdot a_n$ of a nonunit $a \in R$ is ***strongly reduced*** if $a \neq a_1 \cdots \hat{a}_i \cdots a_n$ for any $i \in \{1, \ldots, n\}$

Example: Consider $\mathbb{Q} \times \mathbb{Q}$, then (1,0)=(2,0)($\frac{1}{2}$,0)(2,0)($\frac{1}{2}$,0) is reduced but NOT strongly reduced. Since,

$$(1,0) = (2,0)\widehat{(\frac{1}{2},0)}\widehat{(2,0)}(\frac{1}{2},0)$$

REDUCED UNIQUE FACTORIZATION RINGS

Definition

A commutative ring R is a (***weak***) ***strongly reduced unique factorization ring*** if:
(1) R is atomic, that is, every nonunit of R has a strongly reduced factorization into the product of atoms, and
(2) for every (*nonzero*) nonunit $a \in R$ with $a = a_1 \cdots a_n$ if there exists another strongly reduced factorization $a = b_1 \cdots b_m$ then $n = m$ and after a reordering $a_i \sim b_i$ for $i = 1, \ldots, n$.

Theorem

For a commutative ring R, the following are equivalent:

1. $R[X]$ (weak) strongly reduced UFR
2. $R[X]$ (weak) reduced UFR
3. R is a UFD or a finite direct product $D_1 \times \ldots \times D_n$ where each $n \geq 2$ and D_i is a UFD (possibly a field) where the group of units $U(D_i) = 1$

FACTORIZATION PROPERTIES

Let $R^{\#}$ be the set of nonzero nonunits.

1. **atomic** - each $a \in R^{\#}$ is a product of a finite number of irreducibles (atoms)
2. **Ascending Chain Condition on Principal Ideals (ACCP)** - there does not exist an infinite strictly ascending chain of principal ideals of R
3. **Unique Factorization Ring (UFR)** - every $a \in R^{\#}$ can be written uniquely as the product of irreducibles up to order and associates
4. **Half-Factorial Ring (HFR)** - R is atomic and any two factorizations of $a \in R^{\#}$ into the finite product of irreducibles have the same length

FACTORIZATION PROPERTIES

Let $R^{\#}$ be the set of nonzero nonunits.

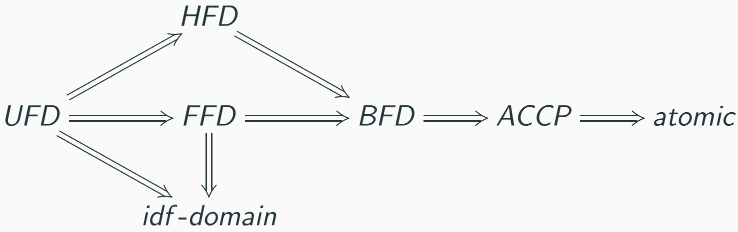
5. **Bounded Factorization Ring (BFR)** - there exists an $N(a)$ for every $a \in R^{\#}$ with $a = a_1 \dots a_n$ where $n \leq N(a)$ and no $a_i \in U(R)$

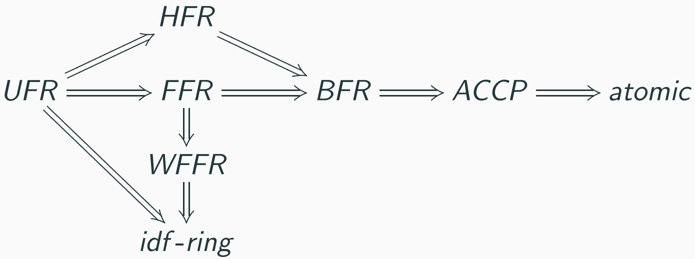
6. **Finite Factorization Ring (FFR)** - every $a \in R^{\#}$ has a finite number of factorizations up to order and associates

7. **Weak Finite Factorization Ring (WFFR)** - every $a \in R$ has a finite number of nonassociate divisors

8. **irreducible- divisor-finite ring (idf-ring)** - every nonzero element of R has at most a finite number of nonassociate irreducible divisors

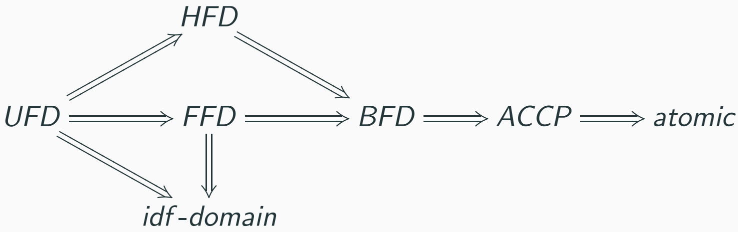
RELATIONSHIP BETWEEN FACTORIZATION PROPERTIES





ASCENSION OF FACTORIZATION PROPERTIES

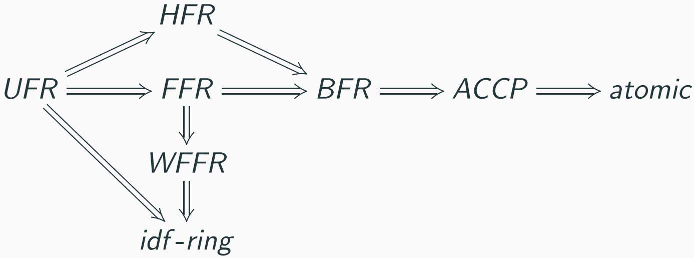
Question: *Which properties ascend from a domain R to $R[X]$?*



Property	UFD	HFD	FFD	idf-domain	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no

ASCENSION OF FACTORIZATION PROPERTIES

Question: *Which properties ascend from a commutative ring R with zero divisors to $R[X]$?*



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

REFERENCES

[1] D.D. Anderson, D. F. Anderson, and Muhammad Zafrullah. *Factorization in integral domains.* Journal of Pure and Applied Algebra 69 (1990), 1-19.

[2] D. D. Anderson and S. Valdes-Leon. *Factorization in commutative rings with zero divisors.* Rocky Mountain Journal of Mathematics 26 (1996), 439-480.

[3] D.D. Anderson and S. Valdes-Leon. *Factorization in Integral Domains, Chapter: Factorization in commutative rings with zero divisors, II.* Publisher: Marcel Dekker, Editors: D.D. Anderson, pp.197-219.

[4] D.D. Anderson and S. Valdes-Leon. *Factorization in commutative rings with zero divisors, III.* Rocky Mountain Journal of Mathematics 31 (2001), 1-22.

REFERENCES

[5] D.D. Anderson and R. Markanda. *Unique factorization rings with zero divisors.* Houston Journal of Mathematics (1985), 15-30.

[6] D.D. Anderson and R. Markanda. *Unique factorization rings with zero divisors: corrigendum.* Houston Journal of Mathematics (1985), 423-426.

[7] W.J. Heinzer and D.C. Lantz. *ACCP in Polynomial Rings: A Counterexample.* Proceedings of the American Mathematical Society 121 (1993), 975-977.

[8] D.D. Anderson, S. Chun, and S. Valdez-Leon. *Reduced Factorization in Commutative Rings with Zero Divisors.* Communications in Algebra 39 (2011), 1583-1594.

CONTACT

Email

Ranthony-Edmonds@uiowa.edu

Personal Website

www.RanthonyEdmonds.com

Slides

www.RanthonyEdmonds.com/conferences-and-presentations.html