GAUSSIAN AMICABLE PAIRS

PATRICK COSTELLO AND RANTHONY A. C. EDMONDS

Abstract. This article defines amicable pairs in the complex numbers and finds that some amicable pairs in the natural numbers are also amicable in the complex numbers. Unlike the case in the natural numbers, it is proved that no $(2, 1)$ pairs made up of natural numbers where the common factor is a power of 2 exist as Gaussian amicable pairs. Many pairs are found with complex parts using the Divisor-Sigma function in *Mathematica*. The factorizations into primes is given so that the type of pair might be determined.

1. Introduction

An *amicable pair* in the natural numbers is a pair of distinct positive integers (a, m) where each integer is the sum of the proper divisors of the other integer. If we let $\sigma(x)$ denote the sum of all divisors of x, then the pair (a, m) is an amicable pair when $\sigma(a) = a + m = \sigma(m)$.

The smallest amicable pair in the natural numbers is the pair $(220, 284)$. Since the function σ is multiplicative, we have $\sigma(220) = \sigma(2^2 \cdot 5 \cdot 11) = \sigma(2^2)\sigma(5)\sigma(11) = 504$. Then the sum of the proper divisors of 220 is $504 - 220 = 284$. This smallest amicable pair was apparently known to Pythagoras.

These numbers have a long and interesting history including discoveries by Pythagoras, Euler, Descartes, and Legendre. See [4] for some of their history. Currently, there are over 1.2 billion pairs known and it is generally believed that there are infinitely many amicable pairs. This has not been proved, but computer searches for new pairs are often successful at finding new pairs.

In this paper, we extend our study of amicable pairs into the realm of complex numbers by looking at Gaussian amicable pairs. The Gaussian integers are the set $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$ and forms a Euclidean domain where there is unique factorization into irreducible elements up to multiplication by units. The irreducible elements are $1 + i$, p a prime in $\mathbb{Z}$ with $p \equiv 3 \pmod 4$, and $a \pm bi$, where $a^2 + b^2$ is a prime q in $\mathbb{Z}$, and where $q \equiv 1 \pmod 4$. The units in $\mathbb{Z}[i]$ are ± 1 and $\pm i$. In [6]

Spira defined the complex sum of divisor function, which extends the sum of divisor function to the Gaussian integers.

Let η be a Gaussian integer with $\eta = \mu \prod \pi_i^{k_i}$, where μ is a unit and each π_i is a Gaussian prime. Let $\sigma^*(\eta)$ denote the complex sum of the divisors function. First, note that $\sigma^*(\eta)$ is multiplicative if we set $\sigma^*(\mu) = 1$ and $\sigma^*(\eta) = \prod \sigma^*(\pi_i^{k_i})$. Here, we have that $\sigma^*(\pi^k) = \sigma^*((-\pi)^k) = \sigma^*((i\pi)^k) = \sigma((-i\pi)^k)$ and thus, independent of the choice of associate of π. Then we define $\sigma^*(\eta)$ as follows,

$$\sigma^*(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}.$$

Note that we require each π_i to lie in the first quadrant of the complex plane, which we take to include the positive half-line and exclude the imaginary half-line. Doing so ensures a unique factorization of η up to order and units. This generalizes the convention of restricting the real-valued sum of divisors function to positive divisors. In fact, this definition extends many of the properties of the real-valued sum of divisors function sigma.

It is known that $\sigma(n) > n$ for any integer $n > 2$. It is then possible to define a *perfect* number, an integer n equal to the sum of its proper divisors, that is $\sigma(n) - n = n$, or $\sigma(n) = 2n$. Similarly, it is shown in [6] that $|\sigma^*(\eta)| \geq |\eta|$ if we require that each associate of η is chosen to lie in the first quadrant. Here, Spira uses the convention that a Gaussian integer η is perfect, if $\sigma^*(\eta) = (1+i)\eta$, and is norm perfect, if $||\sigma^*(\eta)|| = 2||\eta||$. There is also a notion of complex Mersenne numbers. Thus, the complex sum of divisor function can be used to extend the study of many familiar number theoretic properties in the integers to the complex numbers.

We can now define amicable pairs in the Gaussian integers.

Definition 1.1. Two Gaussian integers m and n are *amicable* if $\sigma^*(m) - m = n$ and $\sigma^*(n) - n = m$.

This definition is a natural extension of amicable pairs in the integers. It also has the nice property that a large class of amicable pairs in the natural numbers remain amicable in the Gaussian integers, which is shown in Section 2. Thus, we are convinced it is the correct one. We will also find Gaussian amicable pairs that have complex parts with the methods used outlined in Section 4. In $\mathbb{Z}$, amicable pairs are often organized by type, which refers to the number of primes attached to the common factor of the two numbers. We will show that no rational amicable pairs of type $(2,1)$ with a common factor which is a power of 2 carries over as a Gaussian amicable pair.

The web contains some mention of Gaussian amicable pairs. They are mentioned briefly on Wolfram MathWorld in connection with rational amicable pairs [7]. Sixteen pairs are split into their real and imaginary parts on the On-Line Encyclopedia of Integer Sequences [5], entries A102924 and A102925, respectively.

2. Pairs in $\mathbb{Z}$ that Carry Over

If the integers m and n in $\mathbb{Z}$ are amicable and all the primes involved in both numbers are congruent to $3 \pmod 4$, then these prime factors remain prime in the Gaussian integers. Then the complex sum of divisor function σ^* agrees with σ on all of the primes involved. Consequently, for such a pair m and n in $\mathbb{Z}$, the pair would satisfy the needed condition to be amicable in the Gaussian integers. For example, consider the following 17-digit amicable pair in $\mathbb{Z}$ discovered by Costello in 2003 [2].

$$14435885714987583 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 251 \cdot 2243 \cdot 30911$$
$$14499012954908097 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 11087 \cdot 1576511$$

An inspection of the primes involved in both numbers reveals that all the primes are congruent to $3 \pmod 4$. [Note: When doing this inspection, it is only necessary to look at the last two digits of the prime. If the last two digits are congruent to $3 \pmod 4$, then the prime is congruent to $3 \pmod 4$.] Hence, all the primes in both numbers of this amicable pair remain prime in the Gaussian integers. This pair of numbers then becomes a Gaussian amicable pair. [Note: In $\mathbb{Z}$, this pair is a type $(3, 2)$ pair because the first number contains 3 distinct primes after the common factor $3^4 \cdot 7^2 \cdot 11 \cdot 19$ and the second number contains 2 distinct primes after the common factor.]

In order for m and n in $\mathbb{Z}$ to be amicable and all the primes involved in both numbers be congruent to $3 \pmod 4$, the common factor must not contain a power of 2. The common factor may start with a power of 3 and a power of 7 (as in the pair above), but not include a power of 5. In order for one of the two numbers to be abundant (i.e., the sum of divisors of the proper divisors of the number is greater than the number which is necessary for one of the two numbers), most known odd pairs start with a power of 3 and then contain either a power of 5 or a power of 7. At first, we inspected pairs by hand, but then requested that Sergei Chernykh [2] let the computer do a search for amicable pairs with the condition that all primes were congruent to $3 \pmod 4$. Doing a search in the lists of amicable pairs in $\mathbb{Z}$, the computer was able to find 738,180 known pairs, where all the primes in the two numbers are congruent to $3 \pmod 4$. All of these pairs carry over as Gaussian amicable pairs. A few are listed in Appendix A.

3. Pairs in $\mathbb{Z}$ that Do Not Carry Over

The number 2 is not a prime in the Gaussian integers, but it acts as a part of the common factor in a large number of amicable pairs in $\mathbb{Z}$ including the smallest pair. Consequently, it was decided to determine a formula for $\sigma^*(2^n)$. Computing the σ^* values for the smallest powers of 2, you get the following results.

n	$\sigma^*(2^n)$
1	$2+3i$
2	$-4+5i$
3	$-8-7i$
4	$16-15i$
5	$32+33i$

Looking at these results, a pattern was observed. The rational parts are always $\pm 2^n$ and the coefficient of i is always $\pm(2^n \pm 1)$. The pattern repeats for every group of four n values. The explicit formula for the sequence of signs is summarized in the following theorem.

Theorem 3.1. *If n is a natural number, then*

$$\sigma^*(2^n) = (-1)^{\lfloor n/2 \rfloor} 2^n + (-1)^{\lfloor (n-1)/2 \rfloor}(2^n + (-1)^{\lfloor (n-1)/2 \rfloor})i,$$

where $\lfloor x \rfloor$ is the greatest integer in x.

The proof of this theorem is an induction proof. It hinges on the fact that 2 factors as $2 = (1+i)(1-i)$ and $(1+i)^4(1-i)^4 = (2i)^2 \cdot (-2i)^2 = 2^4$ so 2^{n+4} factors in the same way as 2^n. Details can be found in [3].

This theorem allows us to conclude that pairs of numbers in $\mathbb{Z}$ that are type $(2,1)$ with a power of 2 as the common factor will not exist as Gaussian amicable pairs because the complex sum of divisor function will cause a complex part to be contributed by the power of 2.

Corollary 3.2. *There are no amicable pairs of the form $(2^n pq, 2^n r)$ in $\mathbb{Z}$ with p, q, and r distinct odd primes that are amicable in the Gaussian integers.*

Proof. Suppose that $(2^n pq, 2^n r)$ with p, q, and r primes is an amicable pair of numbers in $\mathbb{Z}$. This pair must satisfy the condition that $\sigma(2^n pq) = \sigma(2^n r)$. Since σ is multiplicative, we get that $\sigma(pq) = \sigma(r)$. But p, q, and r are assumed to be primes and so $(p+1)(q+1) = r+1$. Solving for r gives $r = pq+p+q$. For the two numbers $2^n pq$ and $2^n r$ to be of type $(2,1)$, both p and q must be odd primes, and so congruent to 1 or 3 (mod 4). So the prime r is actually congruent to 3 (mod 4). It remains prime in the

Gaussian integers. Now we compute the complex sum of divisors minus the number $2^n r$. Since r remains prime, we have

$$\sigma^*(2^n r) - 2^n r = \sigma^*(2^n)\sigma^*(r) - 2^n r = \sigma^*(2^n)(r+1) - 2^n r. \qquad (*)$$

For $2^n pq$ and $2^n r$ to be a Gaussian amicable pair, the result in (*) should be $2^n pq$, which has no complex part. However, Theorem 1 says that $\sigma^*(2^n)$ has a complex part. Hence, $2^n pq$ and $2^n r$ cannot be a Gaussian amicable pair. $\square$

4. Searching For Pairs Not in $\mathbb{Z}$

Now we concentrate on finding Gaussian amicable pairs that have a complex part in at least one member of the pair. There are two *Mathematica* commands that become helpful with this search. The command `DivisorSigma`$[k, n]$computes the sum of the kth powers of the divisors of an integer n. Considering the rational integer $n = 6$, the command `DivisorSigma[1,6]` computes that sum $1+2+3+6 = 12$. The importance of this command for us is that one of the options available allows for working with Gaussian integer inputs. The syntax for the command with this option is `DivisorSigma`$[k, n,$ `GaussianIntegers` $\to$ `True`$]$. This will now compute the sum of the kth powers of the divisors of the integer n, which is a Gaussian integer. It was checked with the design team [1] at Wolfram Research that `DivisorSigma`$[1, n,$ `GaussianIntegers` $\to$ `True`$]$computes the same result as the complex divisor function σ^* as defined by Spira [6].

Then the crux for finding Gaussian amicable pairs is the following short test in *Mathematica*:

```
m = DivisorSigma[1, n, GaussianIntegers → True] − n;
x = DivisorSigma[1, m, GaussianIntegers → True] − m;
If [x == n, Print[m, ‘‘ and ’’, n, ‘‘ are amicable’’]]
```

One type of search done in connection with this test was simply to use nested loops that created m values as $m = a + bi$, where a ranged from 0 to a large integer and b ranged from 0 to a large integer. Many pairs were found this way including the pair

$$582448 + 34161i$$
$$70352 - 552561i.$$

Once a pair like this was found, we wished to report its prime factorization in the Gaussian integers just as lists of the amicable pairs in $\mathbb{Z}$ include factorization into primes. The *Mathematica* command `FactorInteger[`n`]` will compute the prime factorization of a rational integer n. Like the

`DivisorSigma` function, this function includes an option for working with Gaussian integers. The syntax for using this option is very similar to the syntax above. We can use

$$\texttt{FactorInteger}[m, \texttt{GaussianIntegers} \rightarrow \texttt{True}]$$

to obtain the prime factorization of m as a Gaussian integer. When we do this on the pair of Gaussian integers above, we get

$$\begin{aligned} 582448 + 34161i &= (1+2i)^3(5+2i)(2+7i)(5+22i) \cdot 59 \\ 70352 - 552561i &= (-i)(1+2i)^3(5+2i)(1+4i)(29+6i)(29+70i). \end{aligned}$$

Ignoring the unit displayed on the second number, this complex amicable pair would be categorized as type $(3,3)$, where the first number has 3 additional primes after the common factor $(1+2i)^3(5+2i)$ and the second number has 3 different additional primes.

Since there exist many amicable pairs in $\mathbb{Z}$ where the common factor is a power of 2, we decided to see if there were complex amicable pairs where the common factor is a power of $(1+i)$, which is the nonunit prime divisor of 2 in the Gaussian integers (2 actually factors as $2 = -i(1+i)^2$). We did searches where the integer m was chosen as $(1+i)^k(a+bi)$ and where k was a small positive integer and a and b were chosen from a large range of integers. Many pairs were found this way including the pair

$$\begin{aligned} 17648 + 768i &= (1+i)^8(1103+48i) \\ 736 - 16560i &= (-i)(1+i)^8 \cdot 23 \cdot (45+2i). \end{aligned}$$

Ignoring the unit displayed on the second number, this complex amicable pair would be categorized as type $(2,1)$, where the second number has 2 additional primes after the common factor $(1+i)^8$ and the first number has 1 different additional prime.

Additional searches were performed with common factors that were powers of $(1+2i)$ and $(1+4i)$, which are the Gaussian primes involved in the rational primes 5 and 17. Pairs of many different types were found ranging from $(2,1)$ pairs to $(5,5)$ pairs. There were also exotic pairs found. (Exotic pairs is the term used in $\mathbb{Z}$ for pairs where the gcd of the two numbers is 1 or common primes have different powers on them in the two numbers, or after the common factor the primes have a power greater than 1.) One exotic Gaussian amicable pair found was

$$\begin{aligned} 301559 - 146012i &= (1+2i)^3(1+4i)(71+16i)(57+82i) \\ -142839 - 241828i &= (1+2i)(1+4i)(7+8i)(13+22i)(16+111i)(-i). \end{aligned}$$

Over 100 Gaussian amicable pairs were found having a complex part. A large percentage (42) pairs of these were $(3, 2)$ pairs. Some of the pairs obtained were listed in Table 1. A complete list of pairs discovered is available upon request to the authors.

TABLE 1.
Gaussian Amicable Pairs with a Complex Part

type	pair	
(2,1)	$736 - 16560i$	$= (1+i)^8(45+2i)(23)(-i)$
	$17648 + 768i$	$= (1+i)^8(1103+48i)$
(2,2)	$-6468 - 5251i$	$= (1+2i)^2(1+4i)(1+14i)\cdot$ $(27+10i)(-i)$
	$5356 - 6133i$	$= (1+2i)^2(1+4i)(3+8i)(36+29i)$
(3,2)	$-14612 - 7159i$	$= (1+2i)^2(6+i)(1+4i)(7+8i)\cdot$ $(10+7i)(-i)$
	$4212 - 19241i$	$= (1+2i)^2(6+i)(1+14i)(23+40i)$
(3,2)	$-4694 + 467i$	$= (1+2i)^2(4+i)(4+5i)(11+34i)(-i)$
	$-766 - 6187i$	$= (1+2i)^2(1+6i)(116+169i)$
(3,2)	$-22233 - 7876i$	$= (1+2i)(1+4i)(2+3i)(5+8i)\cdot$ $(61+44i)(-i)$
	$3033 - 28124i$	$= (1+2i)(1+4i)(9+10i)(5+228i)$
(3,3)	$44764 + 38148i$	$= (1+i)^5(1+2i)^2(1+6i)(23+18i)\cdot$ $(11+4i)(-i)$
	$46436 + 36252i$	$= (1+i)^5(1+2i)^2(5+6i)\cdot$ $(1+14i)(19)(-i)$
(4,2)	$-1105 + 1020i$	$= (1+2i)(2+i)(4+i)(1+4i)\cdot$ $(12+13i)(-i)$
	$-2639 - 1228i$	$= (1+2i)(5+22i)(25+52i)$
(4,3)	$-8547 + 4606i$	$= (1+2i)(2+3i)(1+4i)(29+30i)\cdot$ $(7)(-i)$
	$-8733 - 10366i$	$= (1+2i)(1+6i)(1+14i)(71)$
(5,5)	$2335041 + 13975712i$	$= (1+2i)^3(4+5i)(7+2i)\cdot$ $(5+6i)(1+16i)(150+157i)$
	$15760959 - 1495712i$	$= (1+2i)^3(1+4i)(3+8i)\cdot$ $(14+15i)(26+i)(38+65i)(-i)$
exotic	$-3235 + 1020i$	$= (1+2i)(2+i)^3(5+4i)(7+20i)(-i)$
	$-3549 - 4988i$	$= (1+2i)(1+10i)(15+272i)$

APPENDIX A.

Amicable Pairs in $\mathbb{Z}$ that are also Gaussian Amicable Pairs.

12-digit pairs

te Riele 1995

$$294706414233 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 47 \cdot 7559$$
$$305961592167 = 3^4 \cdot 7 \cdot 11 \cdot 19 \cdot 971 \cdot 2659$$

Einstein&Moews 1996

$$518129600373 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19 \cdot 56783$$
$$523630799307 = 3^2 \cdot 7^2 \cdot 11^2 \cdot 19 \cdot 43 \cdot 12011$$

Einstein&Moews 1996

$$749347913853 = 3^5 \cdot 7 \cdot 11 \cdot 19 \cdot 43 \cdot 49019$$
$$758052380547 = 3^2 \cdot 7 \cdot 11^2 \cdot 19 \cdot 43 \cdot 59 \cdot 2063$$

Einstein&Moews 1996

$$920163589191 = 3^5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 43 \cdot 8599$$
$$964086778809 = 3^2 \cdot 7 \cdot 11^2 \cdot 19 \cdot 43 \cdot 154799$$

13-digit pairs

Ball 1990

$$1692477265941 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 71 \cdot 18287$$
$$1697959925739 = 3^4 \cdot 7 \cdot 11^2 \cdot 19^2 \cdot 179 \cdot 383$$

Einstein 1997

$$2808347861781 = 3^2 \cdot 7^2 \cdot 11^2 \cdot 19 \cdot 23 \cdot 83 \cdot 1451$$
$$2961402044139 = 3^2 \cdot 7^2 \cdot 11^2 \cdot 19 \cdot 503 \cdot 5807$$

Escott 1946

$$3959417614383 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 71 \cdot 179 \cdot 239$$
$$4049489137617 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 1151 \cdot 2699$$

Garcia 1985

$$4400950312143 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 107 \cdot 139 \cdot 227$$
$$4475588004657 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 251 \cdot 13679$$

Garcia 1985

$$9190625896683 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 107 \cdot 131 \cdot 503$$
$$9309948700437 = 3^4 \cdot 7 \cdot 11^2 \cdot 19 \cdot 167 \cdot 42767$$

References

[1] K. Bautista, email to the author, October 16, 2013.

[2] S. Chernykh, *Amicable pairs list*, `https://sech.me/ap/`

[3] R. Clark, *Amicable Gaussian Integers*, M. S. Thesis at Eastern Kentucky University, 2013. (Available online at `https://encompass.eku.edu/etd/158/`).

[4] M. Garcia, J. Pederson, H. J. J. te Riele, *Amicable pairs, A Survey*, Proceedings of the *Conference in Number Theory* in Honor of Professor H. C. Williams, Banff, Canada, May 24–30, 2003. (Available online at `http://oai.cwi.nl/oai/asset/4143/04143D.pdf`).

[5] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

[6] R. Spira, *The complex sum of divisors*, The American Mathematical Monthly, **68**, (1961), 120–124.

[7] E. Weisstein, *Amicable pair*, From MathWorld-A Wolfram Web Resource. `http://mathworld.wolfram.com/AmicablePair.html`.

MSC2010: 11R04, 11A25, 11R27

Key words and phrases: amicable numbers, Gaussian primes, sum of divisors

Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, KY 40475
Email address: pat.costello@eku.edu

Department of Mathematics, The Ohio State University, Columbus, OH 43210
Email address: edmonds.110@osu.edu