Gaussian Amicable Pairs: *"Friendly Imaginary Numbers"*

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Question: What are amicable pairs in the integers, i.e. how do we define "real friendly numbers?"

Sum of Divisors Function

- used to calculate the sum of the positive divisors of a given integer n, denoted $\sigma(n)$
- if d is a divisor of n then, $\sigma(n) = \sum_{d|n} d$

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- **ex.**

$$\begin{aligned} \sigma(12) &= 1 + 2 + 3 + 4 + 6 + 12 \\ &= 28 \end{aligned}$$

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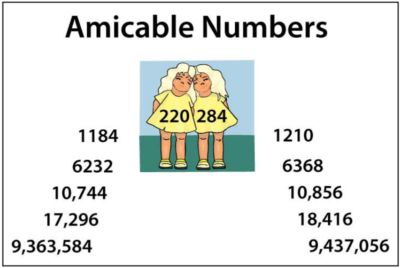
- if p is prime and e is any positive integer $\sigma(p^e) = \dfrac{p^{e+1} - 1}{p - 1}$
- if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$, then $\sigma(n) = \prod_{i=1}^{r} \dfrac{p_i^{(\alpha_i+1)} - 1}{p_i - 1}$

ex.

$$\begin{aligned}\sigma(12) &= \sigma(2^2)\sigma(3)\\ &= \left(\frac{2^{2+1}-1}{2-1}\right)(3+1)\\ &= (7)(4)\\ &= 28\end{aligned}$$



- two integers m and n are said to be *amicable* if $\sigma(m) - m = n$ and $\sigma(n) - n = m$
- proper divisors of one integer equals the proper divisors of the other
- (m, n) is called an *amicable pair*

ex. The smallest amicable pair in $\mathbb{Z}$ is $(220, 284)$

$$\begin{aligned}\sigma(220) &= \sigma(2^2 \cdot 5 \cdot 11)\\ &= \sigma(2^2)\sigma(5)\sigma(11)\\ &= \left(\frac{2^3-1}{2-1}\right)(5+1)(11+1)\\ &= (7)(6)(12)\\ &= 504\end{aligned}$$

and

$$\begin{aligned}\sigma(220) - 220 &= 504 - 220\\ &= 284\end{aligned}$$

$$
\begin{aligned}
\sigma(284) &= \sigma(2^2 \cdot 71) \\
&= \sigma(2^2)\sigma(71) \\
&= \left(\frac{2^3 - 1}{2 - 1}\right)(71 + 1) \\
&= (7)(72) \\
&= 504
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma(284) - 284 &= 504 - 284 \\
&= 220
\end{aligned}
$$

Pairs of a Certain Type

Consider again the pair $(220, 284)$, then $\begin{cases} 220 & = 2^2 \cdot 5 \cdot 11 \\ 284 & = 2^2 \cdot 71 \end{cases}$

So this pair is of the form (Epq, Er) where E is a common factor of both numbers and p, q, and r are distinct primes.

We call pairs of this type $(2, 1)$ pairs.

Pairs of a Certain Type (cont'd)

There are also $(2,2)$ pairs, $(4,3)$ pairs, $(5,1)$ pairs, etc.

Consider the pair $(12285, 14595)$, then $\begin{cases} 12285 & = 3^3 \cdot 5 \cdot 7 \cdot 13 \\ 14595 & = 3 \cdot 5 \cdot 7 \cdot 139 \end{cases}$

We call pairs of this type *erotic* pairs.

Question: What are Gaussian amicable pairs, i.e. how do we define "imaginary friendly numbers?"

Gaussian Integers

- Gaussian integers are denoted $\mathbb{Z}_i$, where $\mathbb{Z}_i = \{a + bi \mid a, b \in \mathbb{Z}\}$

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- units in $\mathbb{Z}_i$ are given by the set: $\{1, -1, i, -i\}$
- Let $p \in \mathbb{Z}_i$ where p is not a unit. The p is *prime* if for every $a, b \in \mathbb{Z}_i$, $p = ab$ implies that either a or b is a unit

Norm in $\mathbb{Z}_i$ and it's Properties

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- $N(z)$ is completely multiplicative, i.e. $N(a)N(b) = N(ab)$
- if $N(z) = 1 \iff z$ is a unit in $\mathbb{Z}_i$
- if $N(z) = p$ where p is prime in $\mathbb{Z}$, then z is prime in $\mathbb{Z}_i$

Complex Sum of Divisors Function

- Let η be a Gaussian integer such that $\eta = \epsilon \prod \pi_i^{k_i}$ where ϵ is a unit and each π_i lies in the first quadrant, then

$$\sigma^\star(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}$$

Amicable Pairs in the Gaussian Integers

- two Gaussian integers m and n are said to be amicable if $\sigma^\star(m) - m = n$ and $\sigma^\star(n) - n = m$

- in order to calculate $\sigma^\star(\eta)$ where $\eta \in \mathbb{Z}_i$ then we must first factor η into its unique factorization up to order and units so that all of the factors of η lie in the first quadrant.

Important Facts

- Let p be an odd prime integer, then p is of the form $4k+1$ or $4k+3$
- If p is of the form $4k+3$, then p is prime in $\mathbb{Z}_i$
- If p is of the form $4k+1$, then p can be written as the sum of squares (i.e. $p = a^2 + b^2$)

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- If p is of the form $4k + 1$, then p can be written as the sum of squares (i.e. $p = a^2 + b^2$)
- if p is an odd prime of the form $4k + 1$ then p can be written as a Gaussian integer $c + di$ where $N(c + di) = p$

Important Facts (cont'd)

- 2^n in $\mathbb{Z}$ factors as $(1+i)^{2n}$ in $\mathbb{Z}_i$
- If the norm of a Gaussian integer z includes a power of 2^n then $(1+i)^n$ is a factor of z

Factoring Gaussian Integers

Consider $-46+20i$. Then we have:

$$
\begin{aligned}
-46+20i &= (1+i)^2(1+4i)(1+6i)(-i) \\
&= (1+i)(1-i)(1+4i)(1+6i) \\
&= (1+i)^2(4-i)(1+6i) \\
&= (1+i)^2(1+4i)(6-i) \\
&= (1+i)^2(-4+i)(1+6i)(-1) \\
&\vdots
\end{aligned}
$$

Factoring Gaussian Integers (cont'd)

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- $N(-46 + 20i) = (-46)^2 + 20^2 = 2516$
- $2516 = 2^2 \cdot 17 \cdot 37$
- this means there are Gaussian integers $a + bi$ and $c + di$ where $N(a + bi) = 17$ and $N(c + di) = 37$

Factoring Gaussian Integers (cont'd)

- In this case we could have $a + bi$ be any of:

$$\{1+4i, 1-4i, -1-4i, -1+4i, 4+i, 4-i, -4-i, -4+i\}$$

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- Need only $1 + 4i$ or $4 + i$

- Similarly, for $c + di$ we use either $1 + 6i$ or $6 + i$.

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- Need only $1 + 4i$ or $4 + i$
- Similarly, for $c + di$ we use either $1 + 6i$ or $6 + i$.
- $-46 + 20i = (1+i)^2(1+4i)(1+6i)(-i)$

Another Factoring Example

Consider: $736 - 16560i$

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- $\dfrac{736 - 16560i}{(1+i)^8} = 46 - 1035i$

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- $\dfrac{46 - 1035i}{23} = 2 - 45i$

Another Factoring Example (cont'd)

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- $\dfrac{2-45i}{45+2i} = -i$

Another Factoring Example (cont'd)

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- $\dfrac{2-45i}{2+45i} = \dfrac{-2021}{2029} - \dfrac{180}{2029}i$
- $\dfrac{2-45i}{45+2i} = -i$
- So $736 - 1650i = (1+i)^8(45+2i)(23)(-i)$

Factoring Gaussian Integers (cont'd)

- needed a way to factor Gaussian Integers efficiently
- developed a Factoring Algorithm
- idea:
  $\rightarrow$ take norm of Gaussian integer
  $\rightarrow$ factor it
  $\rightarrow$ identity if it is a power of $(1+i)$ or is of the form $4k+1$ or $4k+3$
  $\rightarrow$ rewrite factors accordingly
  $\rightarrow$ divide factors out of original Gaussian integer until you are left with a unit

Question: Are there amicable pairs in the integers that are also amicable in the Gaussian integers?

Consider the smallest pair in $\mathbb{Z}$ mentioned above $(220, 284)$, recall

in $\mathbb{Z}$, $\begin{cases} 220 & = 2^2 \cdot 5 \cdot 11 \\ 284 & = 2^2 \cdot 71 \end{cases}$

but

in $\mathbb{Z}_i$, $\begin{cases} 220 & = (1+i)^4(1+2i)(2+i)(11)(i) \\ 284 & = (1+i)^4(71)(-1) \end{cases}$

Applying the complex sum of divisors function, we have:

$$\sigma^{\star}(220) = -672 - 144i$$

and

$$\sigma^{\star}(284) = -288 + 360i$$

So the smallest pair in the integers is <u>not</u> amicable in the Gaussian integers!

Theorem 1. *Let $\sigma^\star$ denote the complex sum of divisors function. Let n be an integer greater than or equal to 1. Then,*

$$\sigma^\star(2^n) = (-1)^{\binom{n+4}{2}} 2^n + (-1)^{\binom{n+3}{2}} (2^n + (-1)^{\binom{n+3}{2}}) i$$

Proof by induction!

This implies that $\sigma^\star(2^n) = x + yi$ where $y \neq 0$.

Theorem 2. *There are no* $(2,1)$ *pairs of the form* $(2^n pq, 2^n r)$ *in* $\mathbb{Z}$ *that are also amicable in* $\mathbb{Z}_i$

The idea is to show $\sigma^{\star}(2^a r) - 2^a r = c + di$ with $d \neq 0$.

Note the relationship between p, q, and r :

$$
\begin{aligned}
r &= (p+1)(q+1) - 1 \\
&= pq + p + q
\end{aligned}
$$

Proof (Case 1): Let $p = 4k + 3$ and $q = 4l + 3$, then

$$\begin{aligned} r &= pq + p + q \\ &= (4k+3)(4l+3) + (4k+3) + (4l+3) \\ &= 4(4kl + 4k + 4l + 3) + 3 \\ &= 4m + 3 \end{aligned}$$

Proof (Case 1): Let $p = 4k + 3$ and $q = 4l + 3$, then

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So we have

$$\begin{aligned} \sigma^\star(2^a r) - 2^a r &= \sigma^\star(2^a)\sigma^\star(r) - 2^a r \\ &= (x + yi)(r+1) - 2^a r \\ &= \big(x(r+1) - 2^a r\big) + y(r+1)i \\ &= c + di \end{aligned}$$

All four cases can be summarized by the following table:

p	q	pq+p+q
4k+1	4k+1	4k+3
4k+3	4k+3	4k+3
4k+1	4k+3	4k+3
4k+3	4k+1	4k+3

So there are no $(2, 1)$ pairs of the form $(2^n pq, 2^n r) \in \mathbb{Z}$ that are also amicable in $\mathbb{Z}_i$

Theorem 3. *Let* (m, n) *be amicable in* $\mathbb{Z}$*. If* $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$ *and* $n = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \ldots \cdot q_s^{\beta_s}$ *where all of the* p_i *and* q_j *are of the form* $4k + 3$*, then* (m, n) *is amicable in* $\mathbb{Z}_i$

Proof.

Consider $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ and $n = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \ldots \cdot q_t^{\beta_t}$. Since each p_i is of the form $4k+3$ the prime factorization of m in the Gaussian integers is the same as its factorization in the integers. But (m, n) is amicable in $\mathbb{Z}$, so:

$$\begin{aligned}\sigma^\star(m) - m &= \sigma(m) - m \\ &= n\end{aligned}$$

and

$$\begin{aligned}\sigma^\star(n) - n &= \sigma(n) - n \\ &= m\end{aligned}$$

Hence (m, n) is also amicable in $\mathbb{Z}_i$. □

Smallest pair satisfying this criteria was discovered by TeRiele in 1995.

$$\begin{cases} 294706414233 = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 47 \cdot 7559 \\ 305961592167 = 3^4 \cdot 7 \cdot 11 \cdot 19 \cdot 971 \cdot 2659 \end{cases}$$

Other examples of Theorem 3

$$\begin{cases} 1111259153519361 & = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 367 \cdot 467 \cdot 587 \\ 1118172210128127 & = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 3023 \cdot 33487 \end{cases}$$

$$\begin{cases} 14435885714987583 & = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 251 \cdot 2243 \cdot 30911 \\ 1449901295908097 & = 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot 11087 \cdot 1576511 \end{cases}$$

$$\begin{cases} 8062452835794819 & = 3^4 \cdot 7^2 \cdot 11^2 \cdot 23 \cdot 71 \cdot 79 \cdot 179 \cdot 727 \\ 8554426893254781 & = 3^4 \cdot 7^2 \cdot 11^2 \cdot 103 \cdot 222 \cdot 479 \cdot 1619 \end{cases}$$

Question: Are there amicable pairs in the Gaussian integers? How do we find "Imaginary Friendly Numbers?"

Formula for 2^n

n	$\sigma^\star(2^n)$
1	2+3i
2	-4+5i
3	-8-7i
4	16-15i
5	32+33i
6	-64+65i
7	-128-127i
8	256-255i
9	512+513i
10	-1024+1025i

Formula for 2^n (cont'd)

- The above table follows the pattern $\pm\sigma^\star(2^n) = \pm 2^n \pm (2^n \pm 1)i$
- The first $\pm$ follows the pattern $+, -, -, +, +, -, -, \ldots$
- The second $\pm$ follows the pattern $+, +, -, -, +, +, -, -, \ldots$
- The patterns in this sequence can be found from Pascal's triangle with binomial coefficients of the form $\binom{k}{2}$ put as exponents on -1

Theorem 1. *Let $\sigma^\star$ denote the complex sum of divisors function. Let n be an integer greater than or equal to 1. Then,*

$$\sigma^\star(2^n) = (-1)^{\binom{n+4}{2}} 2^n + (-1)^{\binom{n+3}{2}} (2^n + (-1)^{\binom{n+3}{2}}) i$$

Proof by induction!

Computer Search for Amicable Pairs in $\mathbb{Z}_i$

- looked for pairs with common factors
- general search
- returned unfactored numbers of the form $a + bi$

Computer Search for Amicable Pairs in $\mathbb{Z}_i$

For $[a = 1, a < 1000000, a++,$Print $[$"$a =$ ", a];
For $[b = 1, b < 100000, b++, x = (1+i)^8 \cdot (a+bi)$;
$y =$ DivisorSigma $[1, x,$ GaussianIntegers $\rightarrow$ True] $-x$;
$z =$ DivisorSigma $[1, y,$ GaussianIntegers $\rightarrow$ True] $-y$;
If $[z == x,$ Print $[x,$ " and " $, y,$" are amicable",
"where the first number has a factor of $(1+i)^8$]]]]

Some Results

$$\begin{cases} -21246 - 8807i = (1+2i)(1+4i)(6+11i)(2+3i)(45+32i)(-i) \\ 5166 - 26953i = (1+2i)(1+4i)(6+11i)(41+234i) \end{cases}$$

$$\begin{cases} 736 - 16560i = (1+i)^8(45+2i)(23)(-i) \\ 17648 + 768i = (1+i)^8(1103+48i) \end{cases}$$

$$\begin{cases} -1036624 + 495520i = (1+i)^8(2+27i)(28+25i)(63+32i) \\ 536656 + 1058336i = (1+i)^8(2+27i)(1055+2528i)(-i) \end{cases}$$

Some Results cont'd

$$\begin{cases} 716246 + 6020977i = (1+2i)(21+10i)(137+180i)(439+270i)(-i) \\ 578954 - 766097i = (1+2i)^2(1+4i)(1+24i)(31+26i)(19+44i)(-i) \end{cases}$$

$$\begin{cases} -6880 + 4275i = (3+2i)^3(1+2i)(2+i)(2+7i)(17+2i)(-i) \\ -8547 + 4606i = (1+2i)(2+3i)(1+4i)(29+30i)(7)(-i) \end{cases}$$

$$\begin{cases} 391696 - 737328i = (1+i)^9(1+2i)^3(2+5i)(1+4i)(147+22i)(-i) \\ 258064 - 702992i = (1+i)^9(1+2i)(2+5i)(1+6i)(3+10i)(28+33i)(- \end{cases}$$

New Amicable Pairs in $\mathbb{Z}_i$ Organized by Type

Type	Number Found
(2,1)	3
(2,2)	19
(3,2)	43
(3,3)	13
(4,2)	3
(4,3)	5
(4,4)	4
(5,3)	4
(5,5)	1
exotic	15
Total	110

New Amicable Pairs in $\mathbb{Z}_i$ Organized by Common Factor

Common Factor	Number Found
$(1+i)^7$	22
$(1+i)^8$	12
$(1+i)^9$	4
$(1+i)^m(1+2i)^n$	5
$(1+2i)$	12
$(1+2i)^2$	15
$(1+2i)^3$	11
$(1+2i)^4$	1
$(1+2i)^m(1+4i)^n$	13
Total	95

Natural Extension: Gaussian aliquot sequences

- Let $s(n) = \sigma(n) - n$, then

$$s^0(n) = n, s^1(n) = s(n), s^2(n) = s(s(n)), \dots$$

  is called an *aliquot sequence*

- classified according to how the sequence terminates (*bounded, amicable, sociable, perfect, aspiring....*)

Natural Extension (cont'd)

- Let $s^\star(n) = \sigma^\star(n) - n$. Then

$$s_0^\star(n) = n, s_1^\star(n) = s^\star(n), s_2^\star(n) = s^\star(s^\star(n)), \ldots$$

is a *Gaussian aliquot sequence*

"The only application or use for these numbers is the original one– you insert a pair of amicable pairs into a pair of amulets, of which you wear one yourself and give the other to your beloved!"

- John Conway

Summary

- criteria for other pairs in $\mathbb{Z}$ that will always carry over to $\mathbb{Z}_i$

- finding pairs of certain types in $\mathbb{Z}_i$

- natural extension: Gaussian aliquot sequences