Unique Factorization in Polynomial Rings with Zero Divisors

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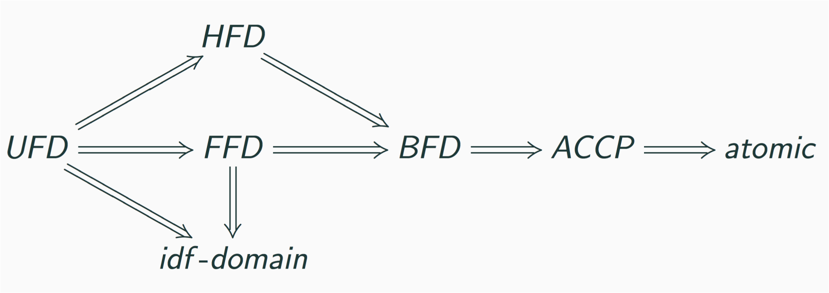
Notation

- R : commutative ring with identity
- $R^{\#}$: set of nonzero nonunits
- $Prin(R)$: set of proper principal ideals
- $Z(R)$: set of zero divisors

Main Goal

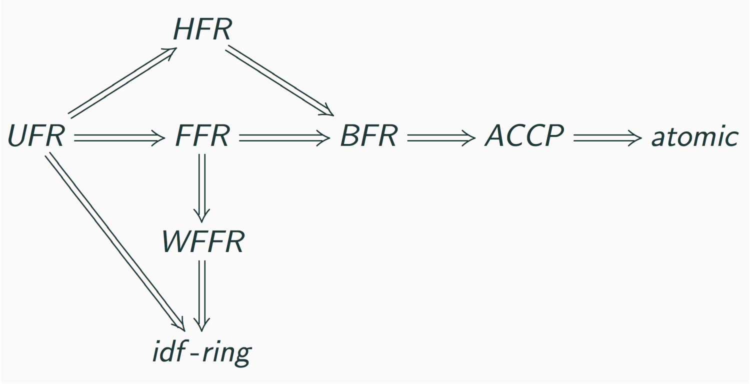
How do certain factorization properties of a commutative ring R behave under the polynomial extension $R[X]$?

Extension of Factorization Properties to $D[X]$



Property	UFD	HFD	FFD	idf-domain	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no

Extenstion of Factorization Properties to $R[X]$



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

Definition

R is a unique factorization ring (UFR) if R is atomic and every $a \in R^{\#}$ can be factored uniquely into the product of atoms up to order and associates

Theorem

Let R be an integral domain. Then R is a UFD $\iff$ $R[X]$ is a UFD

Question: When is $R[X]$ a UFR where R is an arbitrary commutative ring with zero divisors?

Irreducibles in a Domain

Definitions

- $a \in D^{\#}$ is <u>irreducible</u> if $a = bc \implies b \in U(R)$ or $c \in U(R)$
- $a, b \in D^{\#}$ are <u>associated</u>, $a \sim b$, if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$

Theorem (The following are equivalent:)

1. *a is irreducible*
2. $a = bc \implies a \sim b$ *or* $a \sim c$
3. (a) *is maximal in* $Prin(D)$

Note: If $a \in D$ be irreducible, then a is irreducible in $D[X]$

Examples

- $p \in \mathbb{Z}$ is irreducible, and $p \in \mathbb{Z}[X]$ is irreducible
- $a = fg$ in $D[X]$ implies $f, g \in D$

If $a \in R$ be irreducible, then a is not necessarily irreducible in $R[X]$

- 0 is irreducible in K, but (0) is not maximal in $K[X]$, so 0 is not irreducible in $K[X]$

Our notion of irreducible in a domain is too strong in a commutative ring with zero divisors!

Irreducibles in Commutative Rings with Zero Divisors

Types of Associate Relations

associated	$a \sim b$ if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$
strongly associated	$a \approx b$ if $a = ub$ for some $u \in U(R)$
very strongly associated	$a \cong b$ if **(1)** $a \sim b$ **and (2)** $a = b = 0$ or $a \neq 0$ and $a = rb \implies r \in U(R)$

Types of Irreducible Elements

irreducible	$a = bc \implies a \sim b$ or $a \sim c$
strongly irreducible	$a = bc \implies a \approx b$ or $a \approx c$
very strongly irreducible	$a = bc \implies a \cong b$ or $a \cong c$
m-irreducible	(a) is maximal in $Prin(R)$

very strongly associated $\Longrightarrow$ strongly associated $\Longrightarrow$ associated

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prime $\Downarrow$ irreducible

v.s. irreducible $\Longrightarrow$ m-irreducible $\Longrightarrow$ s. irreducible $\Longrightarrow$ irreducible

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p-atomic $\Downarrow$ atomic

v.s. atomic $\Longrightarrow$ m-atomic $\Longrightarrow$ s. atomic $\Longrightarrow$ atomic

Types of UFRs

1. $(\alpha, \beta) - UFR$
2. Bouvier-Galovich UFR
3. Fletcher UFR
4. Reduced UFR
5. Weak UFR

Properties of X

1. X is irreducible $\iff$ R is indecomposable
2. If X is the finite product of n atoms, then R is isomorphic to the finite direct product of n indecomposable rings
3. If X is the finite product of atoms, then the factorization of X is unique

Example

In $\mathbb{Z}_6[X]$, $\qquad X = (3X+2)(2X+3) = 6X^2 + 13X + 6 = X.$

So, $\mathbb{Z}_6[X] \cong R_1[X] \times R_2[X]$ by (2).

Note: $\mathbb{Z}_6[X] \cong \mathbb{Z}_3[X] \times \mathbb{Z}_2[X]$ and $3X+2$ and $2X+3$ are atoms since $3X+2 \mapsto (2, X)$ and $2X+3 \mapsto (2X, 1)$

(α, β)-UFRs

Definition

Let $\alpha \in \{$ atomic, strongly atomic, very strongly atomic, m-atomic, p-atomic $\}$ and $\beta \in \{$ isomorphic, strongly isomorphic, very strongly isomorphic $\}$.

Then R is a <u>(α, β)-unique factorization ring</u> if:

1. R is α
2. any two factorizations of $a \in R^{\#}$ into atoms of the type to define α are β

<u>Note</u>: For any choice of α and β except $\alpha = p$-atomic, R is présimplifiable.

- R is a <u>unique factorization ring</u> if R is an (α, β)-UFR for some (α, β) except $\alpha = p$-atomic.

Bouvier-Galovich UFRs

Bouvier UFR	Galovich UFR
• m-irreducible	• very strongly irreducible
• associate	• strongly associate
• (m-atomic, isomorphic)-UFR	(very strongly atomic, strongly isomorphic)-UFR

Theorem

R is a B-G UFR if R satisfies one of the following:

1. *R is a UFD*
2. (R, M) *is quasi-local where* $M^2 = 0$
3. *R is a special principal ideal ring (SPIR)*

Theorem

$R[X]$ *is a B-G UFR* $\iff$ $R[X]$ *is a UFD*

Bouvier-Galovich UFRs

Theorem

$R[X]$ is a B-G UFR $\iff$ $R[X]$ is a UFD

Proof Sketch.

$\rightarrow$ Let $a, b \in R$ such $ab = 0$ so that a and b are nonzero

$\rightarrow$ $X, X - a$, and $X - b$ are irreducible since R is indecomposable

$\rightarrow$ We have
$$\begin{aligned}(X-a)(X-b) &= X^2 - (a+b)X + ab \\ &= X^2 - (a+b)X \\ &= X(X-(a+b))\end{aligned}$$

$\rightarrow$ A contradiction, so R is a domain and $R[X]$ is a UFD

$\square$

Reduced UFRs

Reduced Factorizations

reduced	$a \neq a_1 \cdots \hat{a}_i \cdots a_n$ for any $i \in \{1, \ldots, n\}$
strongly reduced	$a \neq a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_j} \cdots a_n$ for any nonempty proper subset $\{i_1, \cdots, i_j\} \subsetneq \{1, \ldots, n\}$.

Example

$(1, 0) = (2, 0)(\frac{1}{2}, 0)(2, 0)(\frac{1}{2}, 0)$ in $\mathbb{Q} \times \mathbb{Q}$ is reduced but NOT strongly reduced

Definition

R is a strongly reduced (respectively reduced) UFR if:

1. R is atomic
2. if $a = a_1 \cdots a_n = b_1 \cdots b_m$ are two strongly reduced (respectively reduced) factorizations of a nonunit $a \in R$, then $n = m$ and after a reordering $a_i \sim b_i$ for $i \in \{1, \ldots, n\}$.

Reduced UFRs

Theorem (The following are equivalent:)

1. *$R[X]$ strongly reduced UFR*
2. *$R[X]$ reduced UFR*
3. *R is a UFD or a finite direct product of domains $D_1 \times \cdots \times D_n$ with $n \geq 2$ and each D_i is a UFD (possibly a field) with group of units $U(D_i) = \{1\}$*

Characterizations of other UFRs

Theorem (The following are equivalent:)

1. *$R[X]$ is a Fletcher UFR,*
2. *$R[X]$ is p-atomic,*
3. *R is a finite direct product of UFDs,*
4. *$R[X]$ is factorial, and*
5. *every regular element of $R[X]$ is a product of principal primes*

Theorem (The following are equivalent:)

1. *$R[X]$ is a weak UFR*
2. *every $f \in R[X]^{\#}$ is a product of weakly primes*
3. *$R[X]$ is atomic and each atom is weakly prime*
4. *R is the finite direct product of UFDs*

Main Result

Theorem (Anderson, Edmonds 2018)

$R[X]$ is a UFR if and only if R is a UFD or isomorphic to the finite direct product of UFDs.