Factorization in Polynomial Rings with Zero Divisors

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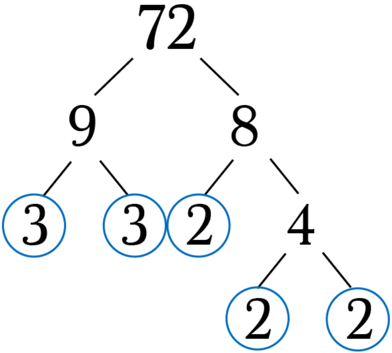
Main Goal

How do certain factorization properties of a commutative ring R behave under the polynomial extension $R[X]$?

Unique Factorization



Fundamental Theorem of Arithmetic (FTA)
every integer can be factored uniquely into the product of primes

Unique Factorization

Unique Factorization Domain
every element can be factored uniquely into the product of atoms

Example

Rings with the Unique Factorization Property

- $\mathbb{Z}$
- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{Z}[X]$
- $\mathbb{Z}/4\mathbb{Z}$

In $\mathbb{Z}/4\mathbb{Z}$, $2 \cdot 2 = 0$ so 2 is called a **zero divisor**

Note: a **domain** is a commutative ring with where 0 is the only zero divisor

Non-Unique Factorization

Consider the ring: $\boxed{\mathbb{R} + X\mathbb{C}[X]}$

<u>in</u>

- $\sqrt{3} + X(2iX^3 + 7X + i)$
- X
- $\left(\frac{1+i}{2}\right)X$

<u>out</u>

- $3i$
- $1 + i$

Factorization of X^2 in $\mathbb{R} + X\mathbb{C}[X]$

$$\begin{aligned} X^2 &= X \cdot X \\ &= (iX)(-iX) \\ &= (1+i)X\left(\frac{1-i}{2}\right)X \\ &= \underbrace{(2+i)X\left(\frac{2-i}{5}\right)X}_{X^2 \text{is divisible by } \{(r+i)X\}} \end{aligned}$$

half-factorial ring: every factorization of a nonzero nonunit element into atoms has the same length

Non-Unique Factorization

finite factorization ring: every nonzero nonunit has only a finite number of factorizations into atoms

Example

Examples of FFRs

- any UFR
- some HFRs, $\mathbb{Z}\sqrt{-5} = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$;

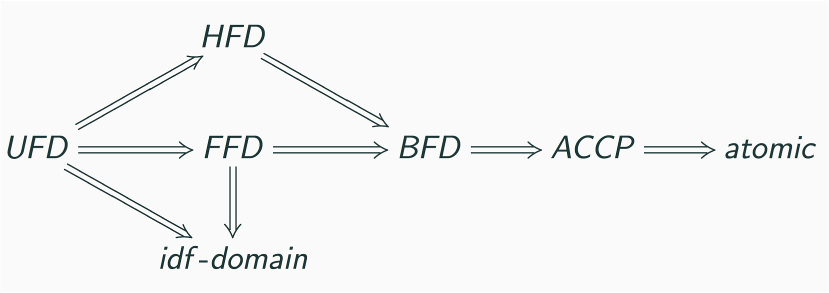
$$6 = 3 \cdot 2 = (1 - \sqrt{-5})(1 + \sqrt{-5})$$

- $\mathbb{R}[X^2, X^3]$

$$X^6 = X^3 \cdot X^3 = X^2 \cdot X^2 \cdot X^2$$

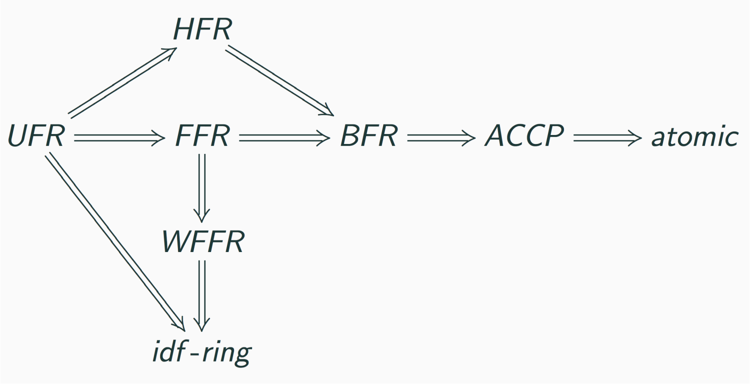
atomic: every nonzero nonunit element can be written as the finite product of atoms

Extension of Factorization Properties to $D[X]$



Property	UFD	HFD	FFD	idf	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no

Extenstion of Factorization Properties to $R[X]$



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

Definition

R is a unique factorization ring (UFR) if R is atomic and every $a \in R^{\#}$ can be factored uniquely into the product of atoms up to order and associates such that if $x = a_1 \cdots a_n = b_1 \cdots b_m$ are two factorizations of nonzero nonunit element x into atoms

1. $n = m$
2. $a_i \sim b_i$ for every i after a reordering

Theorem

Let R be an integral domain. Then R is a UFD $\iff$ $R[X]$ is a UFD

Example

$X^2 = X \cdot X = (X + 2)(X + 2)$ in $\mathbb{Z}/4\mathbb{Z}[X]$

Question: When is $R[X]$ a UFR where R is an arbitrary commutative ring with zero divisors?

Issues

1. Lack of uniformity in the theory
2. Nontrivial idempotents

Definition

We say $e \in R$ is an idempotent if $e^2 = 0$.

- If $e^2 = e$ then $e(e - 1) = 0$.
- $Id(R) = Id(R[X])$

Example

$3 \in \mathbb{Z}_6$ is an atom, $3 = 3, 3 = 3 \cdot 3, 3 = 3 \cdot 3^2, \ldots, 3 = 3^n$

Example

$(1, 0) = (2, 0)(\frac{1}{2}, 0)(2, 0)(\frac{1}{2}, 0)$ in $\mathbb{Q} \times \mathbb{Q}$

Irreducibles in a Domain

Definitions

- $a \in D^{\#}$ is <u>irreducible</u> if $a = bc \implies b \in U(R)$ or $c \in U(R)$
- $a, b \in D^{\#}$ are <u>associated</u>, $a \sim b$, if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$

Theorem (The following are equivalent:)

1. *a is irreducible*
2. $a = bc \implies a \sim b$ *or* $a \sim c$
3. (a) *is maximal in* $Prin(D)$

Irreducibles in Commutative Rings with Zero Divisors

Types of Associate Relations

associated	$a \sim b$ if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$
strongly associated	$a \approx b$ if $a = ub$ for some $u \in U(R)$
very strongly associated	$a \cong b$ if **(1)** $a \sim b$ **and (2)** $a = b = 0$ or $a \neq 0$ and $a = rb \implies r \in U(R)$

Consider $(0,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$,

- $(0,1) \sim (0,1)$ since $<(0,1)> = <(0,1)>$
- $(0,1) \approx (0,1)$ since $(0,1) = (1,1)(0,1)$
- $(0,1) \not\cong (0,1)$ since $(0,1) = (0,1)(0,1)$

Note: We say R is **présimplifiable** if all of the associate conditions agree, i.e. if $x = xy$ implies $x = 0$ or $y \in U(R)$

Irreducibles in Commutative Rings with Zero Divisors

Types of Associate Relations

associated	$a \sim b$ if $a \mid b$ and $b \mid a$, i.e. $(a) = (b)$
strongly associated	$a \approx b$ if $a = ub$ for some $u \in U(R)$
very strongly associated	$a \cong b$ if **(1)** $a \sim b$ **and** **(2)** $a = b = 0$ or $a \neq 0$ and $a = rb \implies r \in U(R)$

Types of Irreducible Elements

irreducible	$a = bc \implies a \sim b$ or $a \sim c$
strongly irreducible	$a = bc \implies a \approx b$ or $a \approx c$
very strongly irreducible	$a = bc \implies a \cong b$ or $a \cong c$
m-irreducible	(a) is maximal in $Prin(R)$

very strongly associated $\Longrightarrow$ strongly associated $\Longrightarrow$ associated

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$$\begin{array}{ccccccc} & & & & & & \text{prime} \\ & & & & & & \Downarrow \\ \text{v.s. irreducible} & \Longrightarrow & m\text{-irreducible} & \Longrightarrow & \text{s. irreducible} & \Longrightarrow & \text{irreducible} \end{array}$$

$$\begin{array}{ccccccc} & & & & & & p\text{-atomic} \\ & & & & & & \Downarrow \\ \text{v.s. atomic} & \Longrightarrow & m\text{-atomic} & \Longrightarrow & \text{s. atomic} & \Longrightarrow & \text{atomic} \end{array}$$

Types of UFRs

1. Fletcher UFR (1969)
2. Bouvier-Galovich UFR (1974-1978)
3. $(\alpha, \beta) - UFR$ (1996)
4. Reduced UFR (2003)
5. Weak UFR (2011)

Properties of X

Theorem (Anderson, Edmonds '18)

Let R be a commutative ring and X an indeterminate over R.

1. *X is irreducible $\iff$ R is indecomposable*
2. *If X is the finite product of n atoms, then R is isomorphic to the finite direct product of n indecomposable rings*
3. *If X is the finite product of atoms, then the factorization of X is unique*

Example

In $\mathbb{Z}_6[X]$, $\qquad X = (3X+2)(2X+3) = 6X^2 + 13X + 6 = X.$

So, $\mathbb{Z}_6[X] \cong R_1[X] \times R_2[X]$ by (2).

Note: $\mathbb{Z}_6[X] \cong \mathbb{Z}_3[X] \times \mathbb{Z}_2[X]$ and $3X+2$ and $2X+3$ are atoms since $3X+2 \mapsto (2, X)$ and $2X+3 \mapsto (2X, 1)$

(α, β)-UFRs

Definition

Let $\alpha \in \{$ atomic, strongly atomic, very strongly atomic, m-atomic, p-atomic $\}$ and $\beta \in \{$ isomorphic, strongly isomorphic, very strongly isomorphic $\}$.

Then R is a (α, β)-unique factorization ring if:

1. R is α
2. any two factorizations of $a \in R^{\#}$ into atoms of the type to define α are β

Note: For any choice of α and β except $\alpha = p$-atomic, R is présimplifiable.

- R is a unique factorization ring if R is an (α, β)-UFR for some (α, β) except $\alpha = p$-atomic.

Bouvier-Galovich UFRs

Bouvier UFR 1974	Galovich UFR 1978
• *m*-irreducible	• very strongly irreducible
• associate	• strongly associate
• (*m*-atomic, isomorphic)-UFR	(very strongly atomic, strongly isomorphic)-UFR

Theorem

R is a B-G UFR if R satisfies one of the following:

1. *R is a UFD*
2. (R, M) *is quasi-local where* $M^2 = 0$
3. *R is a special principal ideal ring (SPIR)*

Theorem

$R[X]$ *is a B-G UFR* $\iff$ $R[X]$ *is a UFD*

Bouvier-Galovich UFRs

Theorem

$R[X]$ is a B-G UFR $\iff$ $R[X]$ is a UFD

Proof Sketch.

$\rightarrow$ Let $a, b \in R$ such $ab = 0$ so that a and b are nonzero

$\rightarrow$ $X, X - a$, and $X - b$ are irreducible since R is indecomposable

$\rightarrow$ We have
$$(X - a)(X - b) = X^2 - (a + b)X + ab$$
$$= X^2 - (a + b)X$$
$$= X(X - (a + b))$$

$\rightarrow$ A contradiction, so R is a domain and $R[X]$ is a UFD

□

Reduced UFRs

Reduced Factorizations

reduced	$a \neq a_1 \cdots \hat{a}_i \cdots a_n$ for any $i \in \{1, \ldots, n\}$
strongly reduced	$a \neq a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_j} \cdots a_n$ for any nonempty proper subset $\{i_1, \cdots, i_j\} \subsetneq \{1, \ldots, n\}$.

Example

$(1,0) = (2,0)(\frac{1}{2},0)(2,0)(\frac{1}{2},0)$ in $\mathbb{Q} \times \mathbb{Q}$ is reduced but NOT strongly reduced

Definition

R is a strongly reduced (respectively reduced) UFR if:

1. R is atomic
2. if $a = a_1 \cdots a_n = b_1 \cdots b_m$ are two strongly reduced (respectively reduced) factorizations of a nonunit $a \in R$, then $n = m$ and after a reordering $a_i \sim b_i$ for $i \in \{1, \ldots, n\}$.

Reduced UFRs

Theorem (Anderson, Edmonds '18)

The following are equivalent:

1. $R[X]$ *strongly reduced UFR*
2. $R[X]$ *reduced UFR*
3. R *is a UFD or a finite direct product of domains* $D_1 \times \cdots \times D_n$ *with* $n \geq 2$ *and each* D_i *is a UFD (possibly a field) with group of units* $U(D_i) = \{1\}$

Note: We need the group of units to be trivial to avoid contradicting that R is strongly reduced.

$$(0, 1, \ldots, 1) = (0, 1, \ldots, 1, u, 1)(0, 1, \ldots, 1, v, 1) = (0, 1, \ldots, 1, \ldots, 1)$$

Fletcher UFRs

Theorem (Anderson, Edmonds '18)

The following are equivalent:

1. *$R[X]$ is a Fletcher UFR,*
2. *$R[X]$ is p-atomic,*
3. *R is a finite direct product of UFDs,*
4. *$R[X]$ is factorial, and*
5. *every regular element of $R[X]$ is a product of principal primes*

Note: Fletcher used U-factorizations to solve problems with nontrivial idempotents

$$3 \in \mathbb{Z}_6 \text{ is an atom}, 3 = 3, 3 = 3 \cdot 3, \ldots, 3 = 3^n$$

$\Rightarrow 3 = 3^n\lceil 3 \rceil$

Weak UFRs

Theorem (Anderson, Edmonds '18)

The following are equivalent:

1. $R[X]$ *is a weak UFR*
2. *every* $f \in R[X]^{\#}$ *is a product of weakly primes*
3. $R[X]$ *is atomic and each atom is weakly prime*
4. R *is the finite direct product of UFDs*

Note: P is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$

Main Result

Theorem (Anderson, Edmonds '18)

$R[X]$ is a UFR if and only if R is a UFD or isomorphic to the finite direct product of UFDs.

Future Directions

- Counterexamples for weaker factorization properties:

$$R(+)N : (r_1, n_1)(r_2, n_2) = (r_1 r_2, r_2 n_1 + r_1 n_2)$$

where $R = D$ a quasi-local domain and $N = D/M$.

Theorem

Let (D, M) be a quasi-local domain with maximal ideal M and let $R = D(+)D/M$, then the following hold:

1. *$R[X]$ satisfies ACCP if and only if R satisfies ACCP*
2. *$R[X]$ is a bounded factorization ring if and only if R is a bounded factorization ring*

$R[X]$ is atomic if and only if R is atomic??????

Future Directions

- Factorization in monoid rings $R[X, M]$:
  "polynomials" in X with coefficients in R and exponents in M

Example

$\mathbb{Z}[X; \mathbb{Z}/2\mathbb{Z}]$ is no longer a domain since
$(X + 1)(X - 1) = X^2 - 1 = 1 - 1 = 0$

Example

$\mathbb{C}[X, \mathbb{Q}^+]$ is an antimatter domain